Eigenvectors and Eigenvalues (Basics)

1 Eigenvalues

Find the eigenvalue for the eigenvector listed below. *(Bonus: Also check that they are eigenvectors.)*

(a) $\begin{bmatrix} 6 & 1 \\ 3 & 4 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 \\ -5 & -6 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$ **(c)** $\begin{bmatrix} 4 & 2 \\ -3 & -3 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ **(d)** $\begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & -6 \\ -1 & 2 & -9 \\ -1 & 2 & -6 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ **(f)** $\begin{bmatrix} 1 & 6 & 9 \\ 0 & 3 & 3 \\ 1 & -1 & 2 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ **(g)** $\begin{bmatrix} -2 & 3 & 1 \\ -7 & 8 & 4 \\ 2 & -2 & 0 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ **(h)** $\begin{bmatrix} 5 & -9 & 2 \\ 8 & -8 & -6 \\ 3 & -3 & -3 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

2 Eigenvectors

Find the eigenvector for the eigenvalue listed below.

(a) $\begin{bmatrix} -2 & -5 \\ 1 & -8 \end{bmatrix}$ $\lambda = -3$ **(b)** $\begin{bmatrix} 2 & -6 \\ -3 & -5 \end{bmatrix}$ $\lambda = -7$ **(c)** $\begin{bmatrix} 1 & -4 \\ 5 & 10 \end{bmatrix}$ $\lambda = 6$ **(d)** $\begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$ $\lambda = 4$

(e) $\begin{bmatrix} -4 & 5 & -7 \\ -3 & 4 & -5 \\ -2 & 1 & 1 \end{bmatrix}$ $\lambda = -3$ **(f)** $\begin{bmatrix} 2 & -2 & 6 \\ 7 & -2 & -4 \\ 2 & -2 & 3 \end{bmatrix}$ $\lambda = 3$ **(g)** $\begin{bmatrix} 1 & -7 & 9 \\ 6 & 9 & 6 \\ 1 & 3 & -1 \end{bmatrix}$ $\lambda = 5$ **(h)** $\begin{bmatrix} -6 & 8 & -1 \\ -7 & 6 & 7 \\ -2 & 2 & 1 \end{bmatrix}$ $\lambda = -1$

3 Characteristic Equations

Find the characteristic equation for the matrices below. *(Bonus: Compute eigenvalues and eigenvectors.)*

(a) $\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 7 & 1 \\ -2 & 4 \end{bmatrix}$ **(c)** $\begin{bmatrix} -1 & 5 \\ 3 & -3 \end{bmatrix}$ **(d)** $\begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 6 & 0 & 1 \\ 0 & -1 & 0 \\ 3 & 0 & 4 \end{bmatrix}$ **(f)** $\begin{bmatrix} 3 & 0 & 2 \\ 0 & -1 & 4 \\ 1 & 0 & 4 \end{bmatrix}$ **(g)** $\begin{bmatrix} -1 & 4 & 2 \\ 0 & -1 & 0 \\ 5 & 0 & 2 \end{bmatrix}$ **(h)** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 4 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

4 Computing Powers Using Eigenvectors and Eigenvalues

The matrix T has $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ with $\lambda_1 = 1$; and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ with $\lambda_2 = -1/2$; and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ with $\lambda_3 = 1/3$. Compute:

(a) $\mathrm{T}^n \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ **(b)** $\mathrm{T}^n \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ **(c)** $\mathrm{T}^n \begin{bmatrix} 0 \\ -4 \\ 4 \end{bmatrix}$ **(d)** $\mathrm{T}^n \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$

(e) $\mathrm{T}^n \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$ **(f)** $\mathrm{T}^n \left(\begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} \right)$ **(g)** $\mathrm{T}^n \left(\begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix} \right)$ **(h)** $\mathrm{T}^n \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

5 MatLab

- We ended last week with the `eig(<matrix>)` command to compute eigenvalues and eigenvectors of a matrix.

```
1 >> [V, D] = eig( [ 1 1 -1; 1 2 1; 1 0 3 ] )

V =                                           D =
    0.8165    0.0000   -0.7071                    1.0000         0         0
   -0.4082   -0.7071    0.0000                         0    3.0000         0
   -0.4082   -0.7071    0.7071                         0         0    2.0000
```

The first output matrix, `V`, contains the eigenvectors as its columns (`V` for eigenVector). For example the first eigenvector of $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ is $\mathbf{v}_1 = \begin{bmatrix} 0.8165 \\ -0.4082 \\ -0.4082 \end{bmatrix}$, which we would probably write as $\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. MatLab divides to make its eigenvectors all have length 1.

The second output matrix, `D`, contains the eigenvalues along its main diagonal (`D` for Diagonal) *in the same order as the eigenvectors in* `V`. For example $\lambda_1 = 1$ is the eigenvalue for $\mathbf{v}_1$ above.

Note that if you do not include "`[V, D] =`" in the command, then MatLab gives only the eigenvalues. MatLab often overloads its functions like this, depending on how the results are caught... this evil can be very confusing.

- To compute the formula for $A^n\mathbf{v}$, say $\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 0 \\ -1 & 2 & 6 \end{bmatrix}^n \begin{bmatrix} 3 \\ -12 \\ -9 \end{bmatrix}$, you first compute the eigenvalues and eigenvectors.

```
2 >> [V, D] = eig( [ 2 0 -1 ; 2 2 0 ; -1 2 6 ] )
```

Then compute the coefficients writing $\mathbf{v}$ in terms of eigenvectors $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots$

```
3 >> c = V \ [ 3 ; -12 ; -9 ]
c =
   -9.7980
    7.3485
  -16.6132
```

```
4 >> c1v1 = c(1) * V(:,1)
c1v1 =
   -4.0000
   -8.0000
    4.0000
```

```
5 >> c2v2 = c(2) * V(:,2)
c2v2 =
    3.0000
   -6.0000
    3.0000
```

```
6 >> c3v3 = c(3) * V(:,3)
c3v3 =
    4.0000
    2.0000
  -16.0000
```

The answer is given by $A^n\mathbf{v} = \lambda_1^n c_1\mathbf{v}_1 + \lambda_2^n c_2\mathbf{v}_2 + \cdots$ which in this case is

$$\begin{bmatrix} 2 & 0 & -1 \\ 2 & 2 & 0 \\ -1 & 2 & 6 \end{bmatrix}^n \begin{bmatrix} 3 \\ -12 \\ 9 \end{bmatrix} = 3^n \begin{bmatrix} -4 \\ -8 \\ 4 \end{bmatrix} + 1^n \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} + 6^n \begin{bmatrix} 4 \\ 2 \\ -16 \end{bmatrix}$$

Note: A slightly slicker way to compute $c_1\mathbf{v}_1$, $c_2\mathbf{v}_2$, etc. is to use a matrix product to perform column operations.

```
7 >> CV = V * diag( V \ [ 3 ; -12 ; -9 ] )
CV =
   -4.0000    3.0000    4.0000
   -8.0000   -6.0000    2.0000
    4.0000    3.0000  -16.0000
```

The MatLab command `diag(<vector>)` creates a matrix which has `<vector>` on the diagonal and `0` everywhere else.
Multiplying on the **right** by this matrix performs **column operations** – multiplying each column of `V` by the corresponding coefficient.